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Homeomorphism flows for non-Lipschitz stochastic differential equations with jumps

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Abstract

In this paper we study the continuity property as well as the homeomorphism property for the solutions of multidimensional stochastic differential equations with jumps and non-Lipschitz coefficients with respect to the initial values.

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1. Introduction and main results

Let $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\geqslant 0})$ be a complete filtered probability space, and $(\mathbb{U}, \mathcal{U}, \nu)$ a σ-finite measurable space. Let $\{W(t)\}_{t\geqslant 0}$ be an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion, and $\{p_t, t \geqslant 0\}$ a stationary $(\mathcal{F}_t)$-Poisson point process with values in $\mathbb{U}$ and with characteristic measure ν(cf. [4]). Let $N_p((0, t], \mathrm{d}u)$ be the counting measure of p_t, i.e., for $A \in \mathcal{U}$

$$N_p((0, t], A) := \#\{0 < s \leqslant t : p_s \in A\},$$

where # denotes the cardinality of a set. The compensator measure of N_p is given by

$$\tilde{N}_p((0, t], \mathrm{d}u) := N_p((0, t], \mathrm{d}u) - t\nu(\mathrm{d}u).$$

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We remark that for $A \in \mathcal{U}$ with $\nu(A) < +\infty$, the random variable $N_p((0,t], A)$ complies with the Poisson distribution with parameter $t\nu(A)$.

In the following, we fix a $\mathbb{U}_0 \in \mathcal{U}$ such that $\nu(\mathbb{U} - \mathbb{U}_0) < \infty$, and consider the following stochastic differential equation (SDE) with jumps in $\mathbb{R}^d$:

$$\begin{aligned} X_t(x) = x &+ \int_0^t b(s, X_s(x))\, \mathrm{d}s + \int_0^t \sigma(s, X_s(x))\, \mathrm{d}W_s \\ &+ \int_0^{t+} \int_{\mathbb{U}_0} f(s, X_{s-}(x), u)\, \tilde{N}_p(\mathrm{d}s, \mathrm{d}u) \\ &+ \int_0^{t+} \int_{\mathbb{U}-\mathbb{U}_0} g(s, X_{s-}(x), u)\, N_p(\mathrm{d}s, \mathrm{d}u), \end{aligned} \tag{1}$$

where $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ satisfy the following assumptions:

($\mathbf{H}_b$) there exists a constant $C_b > 0$ such that for any $s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^d$

$$|b(s,x) - b(s,y)| \leqslant C_b|x-y| \cdot \log(|x-y|^{-1} + \mathrm{e});$$

($\mathbf{H}_\sigma$) there exists a constant $C_\sigma > 0$ such that for any $s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^d$

$$|\sigma(s,x) - \sigma(s,y)|^2 \leqslant C_\sigma|x-y|^2 \cdot \log(|x-y|^{-1} + \mathrm{e});$$

the assumptions on measurable functions f and g from $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{U}$ to $\mathbb{R}^d$ will be specified below. Here, the second integral of the right-hand side in Eq. (1) is taken in the Itô's sense, and the definitions of the third and fourth integrals are referred to [4].

Without the part of jumps in Eq. (1), when b and σ are Lipschitz continuous, it is well known that the unique solution $t \mapsto X_t(\cdot) \in \mathcal{H}(\mathbb{R}^d)$ of Eq. (1) forms a stochastic homeomorphism flow (see, for example [5,9]), where $\mathcal{H}(\mathbb{R}^d)$ denotes the set of all homeomorphism mappings from $\mathbb{R}^d$ to $\mathbb{R}^d$. Recently, the second author in [14,15] proved that this property also holds under ($\mathbf{H}_b$) and ($\mathbf{H}_\sigma$). One of the motivations to study the non-Lipschitz stochastic homeomorphism flows comes from the recent works of Malliavin [8] and Le Jan–Raimond [7] (see also [1,2,11] etc.). The main difficulty to prove the homeomorphism property for non-Lipschitz SDEs is to obtain some moment estimates for negative index by Gronwall's inequality (see Lemma 3.1).

However, when we consider the non-Lipschitz SDEs with jumps, there arise new features. First of all, even in the case of smooth coefficients, the solutions may not form a stochastic homeomorphism flow. A counterexample can be found in [9, p. 328]. The main problem is that there is no restriction on the size of jumps. Secondly, when we follow along the same lines as those in [15], there was no good Novikov type criterion for the exponential martingale in the general case until the recent work of Protter and Shimbo [10]. Nevertheless, in the case that the size of jumps is small and the coefficients are Lipschitz continuous, Fujiwara and Kunita [3] proved that the homeomorphism property for SDEs driven by Lévy processes holds. Later, Kunita in [6] generalized their results to SDEs driven by a general semimartingale. For more discussions about the stochastic diffeomorphism flows, the reader is referred to [5,9].

Our first result is about the continuity of solutions with respect to the initial values. Let $\mathcal{C}(\mathbb{R}^d)$ be the total of all continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^d$. We make the following assumptions on f and g:

($\mathbf{H}_f$) For some $q > (2d) \vee 4$ and any $p \in [2, q]$, there exists a constant $C_p > 0$ such that for any $s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^d$

$$\int_{\mathbb{U}_0} |f(s,x,u) - f(s,y,u)|^p \, \nu(\mathrm{d}u) \leqslant C_p |x-y|^p \cdot \log(|x-y|^{-1} + \mathrm{e}),$$

and

$$\int_{\mathbb{U}_0} |f(s,x,u)|^p \, \nu(\mathrm{d}u) \leqslant C_p (1+|x|)^p.$$

($\mathbf{H}_g$) For each $s \in \mathbb{R}_+$ and $u \in \mathbb{U} - \mathbb{U}_0$, $x \mapsto g(s,x,u) \in \mathcal{C}(\mathbb{R}^d)$.

Under ($\mathbf{H}_b$), ($\mathbf{H}_\sigma$), ($\mathbf{H}_f$) and ($\mathbf{H}_g$), it is well known that there exists a unique strong solution to Eq. (1) (cf. [13]). This solution will be denoted by $X_t(x, \omega)$.

We now state our first result.

Theorem 1.1. *Assume that* ($\mathbf{H}_b$), ($\mathbf{H}_\sigma$), ($\mathbf{H}_f$) *and* ($\mathbf{H}_g$) *hold. Then for almost all* $\omega \in \Omega$, $x \mapsto X_t(x,\omega) \in \mathcal{C}(\mathbb{R}^d)$ *for all* $t \geqslant 0$.

In order to obtain the homeomorphism property, we need the following stronger assumptions on f and g:

($\mathbf{H}'_f$) There exists a positive function $L(u)$ satisfying

$$\sup_{u \in \mathbb{U}_0} L(u) \leqslant \delta < 1 \quad \text{and} \quad \int_{\mathbb{U}_0} L(u)^2 \, \nu(\mathrm{d}u) < +\infty, \tag{2}$$

such that for any $s \in \mathbb{R}_+$, $x, y \in \mathbb{R}^d$ and $u \in \mathbb{U}_0$

$$|f(s,x,u) - f(s,y,u)| \leqslant L(u) \cdot |x-y|,$$

and

$$|f(s,0,u)| \leqslant L(u).$$

Moreover, we also require that for some $q > 4d$

$$\frac{q\delta}{(1-\delta)^{q+1}} < 1. \tag{3}$$

($\mathbf{H}'_g$) For each $s \in \mathbb{R}_+$ and $u \in \mathbb{U} - \mathbb{U}_0$, $x \mapsto x + g(s,x,u) \in \mathcal{H}(\mathbb{R}^d)$.

Remark 1.2. The conditions (2) imply that for any $p \geqslant 2$

$$\int_{\mathbb{U}_0} L(u)^p \, \nu(\mathrm{d}u) < +\infty.$$

The condition (3) is satisfied if we take $\delta = \frac{1}{8d}$. In fact, for $\varepsilon > 0$, let $q = 4d + \varepsilon$ and $f_\varepsilon(x) := (x + 1 + \varepsilon) \cdot \log(\frac{2x-1}{2x})$, then for $x > 1/2$

$$f'_\varepsilon(x) = \frac{x+1+\varepsilon}{x(2x-1)} + \log\left(\frac{2x-1}{2x}\right) > \frac{1}{2x-1} + \log\left(\frac{2x-1}{2x}\right) > 0.$$

Therefore, $x \mapsto f_\varepsilon(x)$ is increasing on $(1/2, +\infty)$, and for ε sufficiently small

$$\frac{4d+\varepsilon}{8d} \leqslant \frac{4+\varepsilon}{8} < \left(\frac{7}{8}\right)^{5+\varepsilon} \leqslant \left(1 - \frac{1}{8d}\right)^{4d+1+\varepsilon} = \mathrm{e}^{f_\varepsilon(4d)}, \quad d \in \mathbb{N}.$$

Our second result is:

Theorem 1.3. *Assume that* $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$, $(\mathbf{H}'_f)$ *and* $(\mathbf{H}'_g)$ *hold. Then for almost all* $\omega \in \Omega$, $x \mapsto X_t(x,\omega) \in \mathcal{H}(\mathbb{R}^d)$ *for all* $t \geqslant 0$.

We now sketch our method. First of all, by a standard stopping time technique as in [3], one may only consider the following SDE with jumps:

$$
\begin{aligned}
Y_t(x) = x &+ \int_0^t b(s, Y_s(x))\,\mathrm{d}s + \int_0^t \sigma(s, Y_s(x))\,\mathrm{d}W_s \\
&+ \int_0^{t+}\int_{\mathbb{U}_0} f(s, Y_{s-}(x), u)\,\tilde{N}_p(\mathrm{d}s, \mathrm{d}u).
\end{aligned} \tag{4}
$$

Secondly, we shall prove that the continuity property of solutions with respect to the initial values as well as the homeomorphism property for Eq. (4) holds in a small time interval by Bihari's inequality. Lastly, by the usual shifting time argument, we extend our results to any large time. It is worth saying that the exponential martingale proved by Protter and Shimbo [10] will play a crucial role in the proof of the homeomorphism property. For the reader's convenience, this as well as another technical lemma in Appendix is presented here. In the next two sections, we shall prove our main results.

The following convention will be used throughout the paper: C with or without indices will denote different positive constants (depending on the indices) whose values may change from one instance to another.

2. Proof of Theorem 1.1

In this section, we suppose that $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$, $(\mathbf{H}_f)$ and $(\mathbf{H}_g)$ hold.

We first explain how to reduce Eq. (1) to Eq. (4). Set $\sigma_0 := 0$, and define for $n \in \mathbb{N}$

$$\sigma_n := \inf\{s > \sigma_{n-1} : p_s \in \mathbb{U} - \mathbb{U}_0\}.$$

Here, we use the convention: $\inf\{\emptyset\} = +\infty$. In view of $\nu(\mathbb{U} - \mathbb{U}_0) < \infty$, for any $T > 0$ the set $\{s \in (0, T] : p_s \in \mathbb{U} - \mathbb{U}_0\}$ only contains a finite number of points, which implies that the stopping time σ_n goes to infinity a.s. as $n \to \infty$.

Let $Y_t(x)$ be the solution of Eq. (4). Set $X_0(x) = x$, and define recursively $X_t(x)$ by

$$
X_t(x) = \begin{cases} Y_{t-\sigma_{n-1}}(X_{\sigma_{n-1}}(x)), & t \in [\sigma_{n-1}, \sigma_n), \\ X_{\sigma_n-}(x) + g(\sigma_n, X_{\sigma_n-}(x), p_{\sigma_n}), & t = \sigma_n. \end{cases} \tag{5}
$$

It is easy to verify that $X_t(x)$ satisfies Eq. (1). From this construction, one finds that it suffices to consider Eq. (4).

We start with the following lemma.

Lemma 2.1. *Suppose that* $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$ *and* $(\mathbf{H}_f)$ *hold. Then, for any* $T > 0$ *and* $p \in [2, \frac{q}{2}]$, *there are two positive constants* $C_1 := C(p, T)$ *and* $C_2 := C(p)$ *such that for any* $x, y \in \mathbb{R}^d$

$$
\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t(x) - Y_t(y)|^p\right) \leqslant C_1\left(|x-y|^p + |x-y|^{p\cdot\exp\{-C_2 T\}}\right), \tag{6}
$$

$$
\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t(x)|^p\right) \leqslant C_1(1+|x|)^p. \tag{7}
$$

Proof. Put

$$Z_t := Y_t(x) - Y_t(y)$$

and

$$F(t,u) := f(t, Y_{t-}(x), u) - f(t, Y_{t-}(y), u).$$

Then

$$\begin{aligned} Z_t &= x - y + \int_0^t (b(s, Y_s(x)) - b(s, Y_s(y)))\, \mathrm{d}s \\ &\quad + \int_0^t (\sigma(s, Y_s(x)) - \sigma(s, Y_s(y)))\, \mathrm{d}W_s + \int_0^{t+} \int_{\mathbb{U}_0} F(s,u)\, \tilde{N}_p(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

By Itô's formula (cf. [4,9]), we have

$$|Z_t|^p = |x-y|^p + A_t^1 + M_t^c + A_t^2 + M_t^d + A_t^3,$$

where

$$\begin{aligned} A_t^1 &:= p \sum_i \int_0^t |Z_s|^{p-2} Z_s^i (b^i(s, Y_s(x)) - b^i(s, Y_s(y)))\, \mathrm{d}s, \\ A_t^2 &:= \frac{p}{2} \int_0^t |Z_s|^{p-2} \cdot |\sigma(s, Y_s(x)) - \sigma(s, Y_s(y))|^2\, \mathrm{d}s \\ &\quad + \frac{p(p-2)}{2} \int_0^t |Z_s|^{p-4} \cdot |(\sigma^*(s, Y_s(x)) - \sigma^*(s, Y_s(y))) Z_s|^2\, \mathrm{d}s, \\ A_t^3 &:= \int_0^{t+} \int_{\mathbb{U}_0} \left(|Z_{s-} + F(s,u)|^p - |Z_{s-}|^p - p \sum_i |Z_{s-}|^{p-2} Z_{s-}^i F^i(s,u) \right) \nu(\mathrm{d}u)\mathrm{d}s, \\ M_t^c &:= p \sum_{i,j} \int_0^t |Z_s|^{p-2} Z_s^i (\sigma^{ij}(s, Y_s(x)) - \sigma^{ij}(s, Y_s(y)))\, \mathrm{d}W_s^j, \\ M_t^d &:= \int_0^{t+} \int_{\mathbb{U}_0} (|Z_{s-} + F(s,u)|^p - |Z_{s-}|^p)\, \tilde{N}_p(\mathrm{d}s, \mathrm{d}u). \end{aligned}$$

Here and after, the asterisk stands for the transpose of a matrix.

For A_t^1 and A_t^2, we have by $(\mathbf{H}_b)$ and $(\mathbf{H}_\sigma)$

$$A_t^1 + A_t^2 \leqslant C \int_0^t |Z_s|^p \log(|Z_s|^{-1} + \mathrm{e})\, \mathrm{d}s.$$

For A_t^3, using Taylor's formula and $(\mathbf{H}_f)$ we have

$$\begin{aligned} A_t^3 &\leqslant C \int_0^t \int_{\mathbb{U}_0} (|F(s,u)|^p + |Z_{s-}|^{p-2} |F(s,u)|^2) \nu(\mathrm{d}u)\mathrm{d}s \\ &\leqslant C \int_0^t |Z_s|^p \log(|Z_s|^{-1} + \mathrm{e})\, \mathrm{d}s. \end{aligned}$$

Thus,

$$h(t) := \mathbb{E}\left(\sup_{s\in[0,t]} |Z_s|^p \right) \leqslant |x-y|^p + C\mathbb{E} \int_0^t |Z_s|^p \log(|Z_s|^{-1} + \mathrm{e})\, \mathrm{d}s$$

$$+\mathbb{E}\left(\sup_{s\in[0,t]}|M_s^c|\right)+\mathbb{E}\left(\sup_{s\in[0,t]}|M_s^d|\right). \tag{8}$$

On the other hand, by Burkholder's inequality, $(\mathbf{H}_\sigma)$ and Young's inequality we have

$$\begin{aligned}\mathbb{E}\left(\sup_{s\in[0,t]}|M_s^c|\right) &\leqslant C\mathbb{E}\left(\int_0^t |Z_s|^{2p}\log(|Z_s|^{-1}+\mathrm{e})\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &\leqslant C\mathbb{E}\left(\sup_{s\in[0,t]}|Z_s|^p\int_0^t |Z_s|^{p}\log(|Z_s|^{-1}+\mathrm{e})\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &\leqslant \frac{1}{4}h(t)+C\mathbb{E}\int_0^t |Z_s|^{p}\log(|Z_s|^{-1}+\mathrm{e})\,\mathrm{d}s. \end{aligned}\tag{9}$$

Noticing that

$$\begin{aligned}Q(s,u) &:= |Z_{s-}+F(s,u)|^p-|Z_{s-}|^p\\ &\leqslant C(|F(s,u)|^p+|Z_{s-}|^{p-1}|F(s,u)|),\end{aligned}$$

we have by Burkholder's inequality (cf. [9, Theorem 48, p. 193])

$$\mathbb{E}\left(\sup_{s\in[0,t]}|M_s^d|\right)\leqslant C\mathbb{E}\left(\int_0^{t+}\int_{\mathbb{U}_0}|Q(s,u)|^2\,N_p(\mathrm{d}s,\mathrm{d}u)\right)^{\frac{1}{2}}\leqslant I_t^1+I_t^2, \tag{10}$$

where

$$\begin{aligned}I_t^1 &:= C\mathbb{E}\left(\int_0^{t+}\int_{\mathbb{U}_0}|F(s,u)|^{2p}\,N_p(\mathrm{d}s,\mathrm{d}u)\right)^{\frac{1}{2}}\\ I_t^2 &:= C\mathbb{E}\left(\int_0^{t+}|Z_{s-}|^{2p-2}\int_{\mathbb{U}_0}|F(s,u)|^{2}\,N_p(\mathrm{d}s,\mathrm{d}u)\right)^{\frac{1}{2}}.\end{aligned}$$

For I_t^1, as in the estimation of (9), we have by Young's inequality and $(\mathbf{H}_f)$

$$\begin{aligned}I_t^1 &= C\mathbb{E}\left(\int_0^{t+}|Z_{s-}|^p\int_{\mathbb{U}_0}\frac{|F(s,u)|^{2p}}{|Z_{s-}|^p}\cdot 1_{\{Z_{s-}\neq 0\}}\,N_p(\mathrm{d}s,\mathrm{d}u)\right)^{\frac{1}{2}}\\ &\leqslant \frac{1}{4}h(t)+C\mathbb{E}\left(\int_0^{t+}\int_{\mathbb{U}_0}\frac{|F(s,u)|^{2p}}{|Z_{s-}|^p}\cdot 1_{\{Z_{s-}\neq 0\}}\,N_p(\mathrm{d}s,\mathrm{d}u)\right)\\ &= \frac{1}{4}h(t)+C\mathbb{E}\left(\int_0^{t+}\left[\int_{\mathbb{U}_0}\frac{|F(s,u)|^{2p}}{|Z_{s-}|^p}\cdot 1_{\{Z_{s-}\neq 0\}}\,\nu(\mathrm{d}u)\right]\mathrm{d}s\right)\\ &\leqslant \frac{1}{4}h(t)+C\mathbb{E}\int_0^t |Z_s|^{p}\log(|Z_s|^{-1}+\mathrm{e})\,\mathrm{d}s. \end{aligned}\tag{11}$$

Similarly, we can estimate

$$I_t^2\leqslant \frac{1}{4}h(t)+C\mathbb{E}\int_0^t |Z_s|^{p}\log(|Z_s|^{-1}+\mathrm{e})\,\mathrm{d}s. \tag{12}$$

Combining (8)–(12) gives that

$$h(t) \leqslant |x-y|^p + \frac{3}{4}h(t) + C\mathbb{E}\int_0^t |Z_s|^p \log(|Z_s|^{-1} + \mathrm{e})\,\mathrm{d}s,$$

where the constant C is independent of t.

Observing that for some $0 < \eta < 1/\mathrm{e}$ and $C_0 > 0$,

$$x^p \cdot \log(x^{-1} + \mathrm{e}) \leqslant C_0 \cdot \rho_\eta(x^p),$$

where ρ_η is a concave and increasing function given by

$$\rho_\eta(x) := \begin{cases} x\log x^{-1}, & x \leqslant \eta, \\ \eta\log\eta^{-1} + (\log\eta^{-1} - 1)(x-\eta), & x > \eta. \end{cases}$$

We further have by Jensen's inequality

$$\begin{aligned} h(t) &\leqslant 4|x-y|^p + C\mathbb{E}\int_0^t |Z_s|^p \log(|Z_s|^{-1} + \mathrm{e})\,\mathrm{d}s \\ &\leqslant 4|x-y|^p + C\mathbb{E}\int_0^t \rho_\eta(|Z_s|^p)\,\mathrm{d}s \\ &\leqslant 4|x-y|^p + C\int_0^t \rho_\eta(\mathbb{E}|Z_s|^p)\,\mathrm{d}s \\ &\leqslant 4|x-y|^p + C\int_0^t \rho_\eta(h(s))\,\mathrm{d}s, \end{aligned}$$

which then gives the desired estimate (6) by Bihari's inequality (cf. [14, Lemma 2.1]).

The second estimate (7) is similar, and we omit the details. □

We now give:

Proof of Theorem 1.1. In Lemma 2.1, because of $p = q/2 > d$ and C_2 being independent of T, we may choose T_0 sufficiently small such that

$$p \cdot \exp\{-C_2 T_0\} > d.$$

Thus, by Kolmogorov's criterion (cf. [3, Lemma 1.1]), there is a càdlàg version of Y on $[0, T_0]$ as a $\mathcal{C}(\mathbb{R}^d)$-valued process, and this version is denoted by $\tilde{Y}$. That is to say, for almost all $\omega \in \Omega$,

$$\text{the mapping } [0, T_0] \ni t \mapsto \tilde{Y}_t(\cdot, \omega) \in \mathcal{C}(\mathbb{R}^d) \text{ is càdlàg.} \tag{13}$$

We now shift the time to obtain the continuity of $x \mapsto Y_t(x)$ for all $t > 0$. Without loss of generality, we assume $t \in [T_0, 2T_0]$. Let $\{Y_t^{T_0}(x), t \in [0, T_0]\}$ be the unique solution of the following SDE:

$$\begin{aligned} Y_t^{T_0}(x) = x &+ \int_0^t b(s+T_0, Y_s^{T_0}(x))\,\mathrm{d}s + \int_0^t \sigma(s+T_0, Y_s^{T_0}(x))\,\mathrm{d}\hat{W}_s \\ &+ \int_0^{t+}\int_{\mathbb{U}_0} f(s+T_0, Y_{s-}^{T_0}(x), u)\,\tilde{N}_{\hat{p}}(\mathrm{d}s, \mathrm{d}u), \end{aligned}$$

where $\hat{W}_s := W_{s+T_0} - W_{T_0}$ and $\hat{p}_s := p_{s+T_0}$.

If we define

$$\tilde{Y}_{t+T_0}(x) := \tilde{Y}_t^{T_0}(\tilde{Y}_{T_0}(x)),$$

then by (13) and the uniqueness of solution, the mapping $[0, T_0] \ni t \mapsto \tilde{Y}_{t+T_0}(\cdot) \in \mathcal{C}(\mathbb{R}^d)$ is a càdlàg version of $Y_{t+T_0}(x)$. The proof is thus complete.

3. Proof of Theorem 1.3

In this section we suppose $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$, $(\mathbf{H}'_f)$ and $(\mathbf{H}'_g)$ hold. As in Section 2, we may assume that $g = 0$ and only consider Eq. (4).

We begin with the following crucial lemma.

Lemma 3.1. *Assume* $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$ *and* $(\mathbf{H}'_f)$. *Then, there exist a time* $T_1 = T(q, \delta) > 0$ *and two positive constants* $C'_1 := C(q, T_1)$ *and* $C'_2 := C(q, \delta)$ *such that for any* $x, y \in \mathbb{R}^d$

$$\mathbb{E}\left(\sup_{t\in[0,T_1]} |Y_t(x) - Y_t(y)|^{-q}\right) \leqslant C'_1(|x-y|^{-C'_2} + 1), \tag{14}$$

where q *and* δ *are from* $(\mathbf{H}'_f)$.

Proof. Set

$$Z_t := Y_t(x) - Y_t(y)$$

and

$$F(t, u) := f(t, Y_{t-}(x), u) - f(t, Y_{t-}(y), u).$$

Then, by $(\mathbf{H}'_f)$ we have

$$|F(t, u)| \leqslant L(u) \cdot |Z_{t-}| \leqslant \delta \cdot |Z_{t-}|. \tag{15}$$

For any $\varepsilon \in (0, |x - y|)$, define the stopping time

$$\tau_\varepsilon := \inf\{t > 0 : |Z_t| < \varepsilon\}.$$

By Itô's formula we have

$$|Z_{t\wedge\tau_\varepsilon}|^{-q} = |x-y|^{-q} + \int_0^{(t\wedge\tau_\varepsilon)+} |Z_{s-}|^{-q}\, \mathrm{d}(A_s + M_s),$$

where $A_s = A^1_s + A^2_s + A^3_s$ is a local finite variation process with

$$A^1_s := -q \sum_i \int_0^s |Z_r|^{-2} Z^i_r (b^i(r, Y_r(x)) - b^i(r, Y_r(y)))\mathrm{d}r$$

$$\begin{aligned} A^2_s := \int_0^s \Bigg(& -\frac{q}{2}|Z_r|^{-2}|\sigma(r, Y_r(x)) - \sigma(r, Y_r(y))|^2 \\ & + \frac{q(q+2)}{2}|Z_r|^{-4}|(\sigma^*(r, Y_r(x)) - \sigma^*(r, Y_r(y)))Z_r|^2\Bigg)\mathrm{d}r \end{aligned}$$

$$\begin{aligned} A^3_s := \int_0^s \Bigg(& |Z_{r-}|^q \int_{\mathbb{U}_0} \Bigg(|Z_{r-} + F(r, u)|^{-q} - |Z_{r-}|^{-q} \\ & + q \sum_i |Z_{r-}|^{-q-2} Z^i_{r-} F^i(r, u)\Bigg) \nu(\mathrm{d}u)\Bigg) \mathrm{d}r, \end{aligned}$$

and $M_s = M_s^c + M_s^d$ is a local martingale with

$$M_s^c := -q \sum_{i,j} \int_0^s |Z_r|^{-2} Z_r^i (\sigma^{ij}(r, Y_r(x)) - \sigma^{ij}(r, Y_r(y))) \, \mathrm{d}W_r^j,$$

$$M_s^d := \int_0^{s+} \int_{\mathbb{U}_0} |Z_{r-}|^q (|Z_{r-} + F(r,u)|^{-q} - |Z_{r-}|^{-q}) \, \tilde{N}_p(\mathrm{d}r, \mathrm{d}u).$$

By $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$ and $(\mathbf{H}'_f)$, we know that

$$t \mapsto A_{t\wedge\tau_\varepsilon} + M_{t\wedge\tau_\varepsilon}$$

is a semimartingale. So, by [9, Theorem 37, p. 84]

$$|Z_{t\wedge\tau_\varepsilon}|^{-q} = |x-y|^{-q} \exp\{A_{t\wedge\tau_\varepsilon}\} \cdot \mathcal{E}(M)_{t\wedge\tau_\varepsilon},$$

where $\mathcal{E}(M)_t$ satisfies $\mathcal{E}(M)_{t\wedge\tau_\varepsilon} = 1 + \int_0^{(t\wedge\tau_\varepsilon)+} \mathcal{E}(M)_{s-} \mathrm{d}M_s$ and is given by

$$\mathcal{E}(M)_{t\wedge\tau_\varepsilon} := \exp\left\{M_{t\wedge\tau_\varepsilon} - \frac{1}{2}[M^c, M^c]_{t\wedge\tau_\varepsilon}\right\} \times \prod_{0<s\leqslant t\wedge\tau_\varepsilon} (1 + \Delta M_s) \mathrm{e}^{-\Delta M_s},$$

where $\Delta M_s := M_s - M_{s-}$.

Put

$$h(t) := \mathbb{E}\left(\sup_{s\in[0,t]} |Z_{s\wedge\tau_\varepsilon}|^{-q}\right),$$

and fix an $\alpha > 1$ (its value will be determined in Lemma 3.3). By Hölder's inequality with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and Young's inequality, we have

$$\begin{aligned}
h(t) &\leqslant |x-y|^{-q} \cdot \mathbb{E}\left(\sup_{s\in[0,t]} \exp\{A_{s\wedge\tau_\varepsilon}\} \cdot \sup_{s\in[0,t]} \mathcal{E}(M)_{s\wedge\tau_\varepsilon}\right) \\
&\leqslant |x-y|^{-q} \cdot \left[\mathbb{E}\left(\sup_{s\in[0,t]} \exp\{\beta A_{s\wedge\tau_\varepsilon}\}\right)\right]^{\frac{1}{\beta}} \left[\mathbb{E}\left(\sup_{s\in[0,t]} \mathcal{E}(M)^\alpha_{s\wedge\tau_\varepsilon}\right)\right]^{\frac{1}{\alpha}} \\
&\leqslant C|x-y|^{-\beta q} \cdot \mathbb{E}\left(\sup_{s\in[0,t]} \exp\{\beta A_{s\wedge\tau_\varepsilon}\}\right) + \mathbb{E}\left(\sup_{s\in[0,t]} \mathcal{E}(M)^\alpha_{s\wedge\tau_\varepsilon}\right) \\
&\leqslant C|x-y|^{-2\beta q} + \mathbb{E}\left(\sup_{s\in[0,t]} \exp\{2\beta A_{s\wedge\tau_\varepsilon}\}\right) + \mathbb{E}\left(\sup_{s\in[0,t]} \mathcal{E}(M)^\alpha_{s\wedge\tau_\varepsilon}\right).
\end{aligned}$$

Using Lemmas 3.2 and 3.3, we further have

$$\begin{aligned}
h(t) &\leqslant C|x-y|^{-2\beta q} + C\mathbb{E}\int_0^t |Z_{s\wedge\tau_\varepsilon}|^{-q} \, \mathrm{d}s + C \\
&\leqslant C|x-y|^{-2\beta q} + C\int_0^t h(s) \, \mathrm{d}s + C,
\end{aligned}$$

where $t \in [0, T_2 \wedge T_3]$, T_2 and T_3 are from Lemmas 3.2 and 3.3 respectively.

By Gronwall's inequality, we obtain for $T_1 := T_2 \wedge T_3$

$$\mathbb{E}\left(\sup_{s\in[0,T_1]} |Z_{s\wedge\tau_\varepsilon}|^{-q}\right) = h(T_1) \leqslant C|x-y|^{-2\beta q} + C,$$

where the constant C is independent of ε. By Fatou's lemma, we obtain

$$\mathbb{E}\left(\sup_{s\in[0,T_1\wedge\tau_0)}|Z_s|^{-q}\right)\leqslant C|x-y|^{-2\beta q}+C, \tag{16}$$

where $\tau_0 := \inf\{t>0 : |Z_t|=0\}$. Noting that by (15)

$$|Z_{t-}|-|Z_t|\leqslant|\Delta Z_t|=|F(t,p(t))|\cdot 1_{\mathbb{U}_0}(p_t)\leqslant\delta\cdot|Z_{t-}|,$$

we have $|Z_t|^{-1}\leqslant|Z_{t-}|^{-1}/(1-\delta)$, which together with (16) and $Z_{\tau_0}=0$ produces by contradiction that $\tau_0>T_1$ a.s., and (14) holds. □

Lemma 3.2. *For any $\beta>1$, there exist a time $T_2=T(\beta,q)>0$ and a constant $C=C(T_2,\beta,q)>0$ independent of ε such that for all $t\in[0,T_2]$*

$$\mathbb{E}\left(\sup_{s\in[0,t]}\exp\{\beta A_{s\wedge\tau_\varepsilon}\}\right)\leqslant C\int_0^t\mathbb{E}|Z_{s\wedge\tau_\varepsilon}|^{-q}\mathrm{d}s+C.$$

Proof. Recalling that $A_s=A_s^1+A_s^2+A_s^3$, by $(\mathbf{H}_b)$ and $(\mathbf{H}_\sigma)$ we have

$$|\beta A^1_{s\wedge\tau_\varepsilon}|+|\beta A^2_{s\wedge\tau_\varepsilon}|\leqslant C_0q\int_0^{s\wedge\tau_\varepsilon}\log(|Z_r|^{-1}+\mathrm{e})\mathrm{d}r.$$

For A_s^3, noticing that by Taylor's formula and (15)

$$\begin{aligned}&\left||Z_{r-}+F(r,u)|^{-q}-|Z_{r-}|^{-q}+q\sum_i|Z_{r-}|^{-q-2}Z^i_{r-}F^i(r,u)\right|\\&\quad\leqslant\frac{C|F(r,u)|^2}{||Z_{r-}|-|F(r,u)||^{q+2}}\leqslant\frac{C\cdot L(u)^2}{|Z_{r-}|^q(1-L(u))^{q+2}}\leqslant C\cdot L(u)^2\cdot|Z_{r-}|^{-q},\end{aligned}$$

we have

$$|\beta A^3_{s\wedge\tau_\varepsilon}|\leqslant C\int_0^{s\wedge\tau_\varepsilon}\int_{\mathbb{U}_0}L(u)^2\,\nu(\mathrm{d}u)\mathrm{d}r\leqslant Cs.$$

Hence

$$\mathbb{E}\left(\sup_{s\in[0,t]}\exp\{\beta A_{s\wedge\tau_\varepsilon}\}\right)\leqslant\mathrm{e}^{Ct}\cdot\mathbb{E}\exp\left\{C_0q\int_0^{t\wedge\tau_\varepsilon}\log(|Z_s|^{-1}+\mathrm{e})\,\mathrm{d}s\right\}.$$

Letting $T_2:=\frac{1}{C_0}$, then by Jensen's inequality we have for $t\in[0,T_2]$

$$\begin{aligned}\exp\left\{C_0q\int_0^{t\wedge\tau_\varepsilon}\log(|Z_s|^{-1}+\mathrm{e})\,\mathrm{d}s\right\}&\leqslant\exp\left\{\frac{1}{T_2}\int_0^{T_2}\log(|Z_s|^{-1}+\mathrm{e})^{q\cdot1_{\{s<t\wedge\tau_\varepsilon\}}}\,\mathrm{d}s\right\}\\&\leqslant\frac{1}{T_2}\int_0^{T_2}(|Z_s|^{-1}+\mathrm{e})^{q\cdot1_{\{s<t\wedge\tau_\varepsilon\}}}\,\mathrm{d}s\\&\leqslant\frac{1}{T_2}\int_0^{T_2}(|Z_s|^{-1}+\mathrm{e})^{q}\cdot1_{\{s<t\wedge\tau_\varepsilon\}}\,\mathrm{d}s+1\end{aligned}$$

$$\leqslant \frac{1}{T_2}\int_0^t (|Z_{s\wedge\tau_\varepsilon}|^{-1}+\mathrm{e})^q\,\mathrm{d}s+1$$
$$\leqslant \frac{2^{q-1}}{T_2}\int_0^t |Z_{s\wedge\tau_\varepsilon}|^{-q}\,\mathrm{d}s+C.$$

The proof is thus complete. □

Lemma 3.3. *For some $\alpha > 1$, there exist a time $T_3 = T(\alpha) > 0$ and a constant $C = C(T_3,\alpha) > 0$ independent of ε such that for any $t \in [0, T_3]$*

$$\mathbb{E}\left(\sup_{s\in[0,t]} \mathcal{E}(M)^\alpha_{s\wedge\tau_\varepsilon}\right) \leqslant C\int_0^t \mathbb{E}|Z_{s\wedge\tau_\varepsilon}|^{-q}\mathrm{d}s + C.$$

Proof. Noticing the following elementary inequality

$$|(1+a)^{-q}-1| \leqslant \frac{q\cdot|a|}{(1-|a|)^{q+1}},\quad a\in(-1,1) \tag{17}$$

and

$$\Delta M_t = |Z_{t-}|^q\cdot(|Z_{t-}+F(t,p_t)|^{-q}-|Z_{t-}|^{-q})\cdot 1_{\mathbb{U}_0}(p_t),$$

we have by (3) and (15)

$$|\Delta M_t| \leqslant \frac{qL(p_t)\cdot 1_{\mathbb{U}_0}(p_t)}{(1-L(p_t))^{q+1}} \leqslant \frac{q\delta}{(1-\delta)^{q+1}} < 1. \tag{18}$$

Moreover, we have by $(\mathbf{H}_\sigma)$

$$\langle M^c, M^c\rangle_{t\wedge\tau_\varepsilon} \leqslant C\int_0^{t\wedge\tau_\varepsilon} \log(|Z_s|^{-1}+\mathrm{e})\,\mathrm{d}s \leqslant C_\varepsilon\cdot t, \tag{19}$$

and by (15) and (17)

$$\langle M^d, M^d\rangle_{t\wedge\tau_\varepsilon} = \int_0^{t\wedge\tau_\varepsilon}\int_{\mathbb{U}_0}\left(\frac{|Z_{r-}+F(r,u)|^{-q}-|Z_{r-}|^{-q}}{|Z_{r-}|^{-q}}\right)^2 \nu(\mathrm{d}u)\mathrm{d}r$$
$$\leqslant C\int_0^{t\wedge\tau_\varepsilon}\int_{\mathbb{U}_0} L^2(u)\,\nu(\mathrm{d}u)\mathrm{d}r \leqslant C\cdot t.$$

Now by (18), we can choose α,γ such that $1<\alpha<\gamma$

$$\frac{\gamma\cdot q\delta}{(1-\delta)^{q+1}} < 1. \tag{20}$$

Thus, by Theorem A.1 in Appendix we know that $t\mapsto \mathcal{E}(M)_{t\wedge\tau_\varepsilon}$ and $t\mapsto \mathcal{E}(\gamma M)_{t\wedge\tau_\varepsilon}$ are two martingales. And so, by Doob's maximal inequality (cf. [9]),

$$\mathbb{E}\left(\sup_{s\in[0,t]} \mathcal{E}(M)^\alpha_{s\wedge\tau_\varepsilon}\right) \leqslant C\mathbb{E}\mathcal{E}(M)^\alpha_{t\wedge\tau_\varepsilon}. \tag{21}$$

We now estimate the right-hand side of (21). Observing that

$$\mathcal{E}(M)^\alpha_{t\wedge\tau_\varepsilon} = \mathcal{E}(\gamma M)^{\alpha/\gamma}_{t\wedge\tau_\varepsilon}\cdot\exp\left\{\frac{(\gamma-1)\alpha}{2}[M^c,M^c]_{t\wedge\tau_\varepsilon}\right\}\cdot\prod_{0<s\leqslant t\wedge\tau_\varepsilon} G(\Delta M_s),$$

where

$$G(x) := \frac{(1+x)^\alpha}{(1+\gamma x)^{\frac{\alpha}{\gamma}}}, \quad |x| \leqslant \frac{q\delta}{(1-\delta)^{q+1}},$$

we have by Hölder's inequality and Young's inequality

$$\begin{aligned}
\mathbb{E}(\mathcal{E}(M)^\alpha_{t\wedge\tau_\varepsilon}) &\leqslant \left(\mathbb{E}\mathcal{E}(\gamma M)_{t\wedge\tau_\varepsilon}\right)^{\frac{\alpha}{\gamma}} \\
&\quad\times \left[\mathbb{E}\left(\exp\left\{\frac{(\gamma-1)\alpha}{2}\cdot\frac{\gamma}{\gamma-\alpha}[M^c,M^c]_{t\wedge\tau_\varepsilon}\right\}\cdot\prod_{0<s\leqslant t\wedge\tau_\varepsilon} G(\Delta M_s)^{\frac{\gamma}{\gamma-\alpha}}\right)\right]^{\frac{\gamma-\alpha}{\gamma}} \\
&\leqslant \mathbb{E}\exp\{C_{\alpha,\gamma}[M^c,M^c]_{t\wedge\tau_\varepsilon}\} + \mathbb{E}\left[\prod_{0<s\leqslant t\wedge\tau_\varepsilon} G(\Delta M_s)^{\frac{2\gamma}{\gamma-\alpha}}\right] + C. \qquad (22)
\end{aligned}$$

By (19) and using the same trick as in the proof of Lemma 3.2, there exists a time $T_3 = T(\alpha,\gamma)$ such that for any $t \in [0, T_3]$

$$\mathbb{E}\exp\{C_{\alpha,\gamma}[M^c,M^c]_{t\wedge\tau_\varepsilon}\} \leqslant C\int_0^t \mathbb{E}|Z_{s\wedge\tau_\varepsilon}|^{-q}\mathrm{d}s + C. \qquad (23)$$

Thanks to (20) and the following limit

$$\lim_{x\downarrow 0}\frac{\log G(x)}{x^2} = \frac{\alpha(\gamma-1)}{2},$$

we have for some $C = C(q,\delta,\alpha,\gamma) > 0$

$$|\log G(x)| \leqslant C|x|^2, \quad \forall |x| \leqslant \frac{q\delta}{(1-\delta)^{q+1}}.$$

Therefore, by (18) we obtain

$$\begin{aligned}
\mathbb{E}\left[\prod_{0<s\leqslant t\wedge\tau_\varepsilon} G(\Delta M_s)^{\frac{2\gamma}{\gamma-\alpha}}\right] &= \mathbb{E}\exp\left\{\sum_{0<s\leqslant t\wedge\tau_\varepsilon}\frac{2\gamma}{\gamma-\alpha}\log G(\Delta M_s)\right\} \\
&\leqslant \mathbb{E}\exp\left\{\sum_{0<s\leqslant t\wedge\tau_\varepsilon} C|\Delta M_s|^2\right\} \\
&\leqslant \mathbb{E}\exp\left\{\sum_{0<s\leqslant t} CL(p_s)^2\cdot 1_{\mathbb{U}_0}(p_s)\right\}. \qquad (24)
\end{aligned}$$

The proof is thus completed by (21)–(24) and Lemma A.2 in Appendix. □

The proof of the following lemma is standard and simpler than Lemma 3.1, so we omit the details.

Lemma 3.4. *Under* $(\mathbf{H}_b)$, $(\mathbf{H}_\sigma)$ *and* $(\mathbf{H}'_f)$, *for any* $T > 0$ *and* $p \geqslant 2$ *it holds that*

$$\mathbb{E}\left(\sup_{t\in[0,T]}(1+|Y_t(x)|^2)^{-p}\right) \leqslant C(1+|x|^2)^{-p}, \quad x \in \mathbb{R}^d. \qquad (25)$$

We are now in a position to give the proof of Theorem 1.3.

Proof of Theorem 1.3. For $T > 0$ and $x \neq y \in \mathbb{R}^d$, define

$$R_i^T(x, y) := \sup_{t\in[0,T]} |Y_t(x) - Y_t(y)|^i, \quad i = 1, -1.$$

By Lemmas 2.1 and 3.1, $R_i^T(x, y)$ are well defined if $T < T_1$.

For any $x, y, x', y' \in \mathbb{R}^d$ with $x \neq y, x' \neq y'$, by an elementary calculation we have

$$|R_{-1}^T(x, y) - R_{-1}^T(x', y')| \leqslant R_{-1}^T(x, y) \cdot R_{-1}^T(x', y') \cdot [R_1^T(x, x') + R_1^T(y, y')].$$

Put $p := \frac{q}{4} + d$ and $\lambda := \frac{2q}{q+4d}$. Then

$$p > 2d, \quad \lambda > 1, \quad 2p\lambda = q.$$

By Hölder's inequality with $\frac{1}{\lambda} + \frac{1}{\kappa} = 1$, we have

$$\begin{aligned}\mathbb{E}&\left|R_{-1}^T(x, y) - R_{-1}^T(x', y')\right|^p \\ &\leqslant 2^{p-1}\mathbb{E}(R_{-1}^T(x, y)^p \cdot R_{-1}^T(x', y')^p \cdot [R_1^T(x, x')^p + R_1^T(y, y')^p]) \\ &\leqslant 2^{p-1}\|R_{-1}^T(x, y)\|_{L^{2p\lambda}}^p \cdot \|R_{-1}^T(x', y')\|_{L^{2p\lambda}}^p [\|R_1^T(x, x')\|_{L^{p\cdot\kappa}}^p + \|R_1^T(y, y')\|_{L^{p\cdot\kappa}}^p],\end{aligned}$$

where $\|\cdot\|_{L^p}$ is the usual norm in $L^p(\Omega, \mathcal{F}, P)$.

By Remark 1.2, we know that the estimates (6) and (7) hold for any $p \geqslant 2$. So, by Lemma 3.1 and (6) we have for any $T \leqslant T_1$

$$\begin{aligned}\mathbb{E}|R_{-1}^T(x, y) - R_{-1}^T(x', y')|^p \leqslant\ & C(|x - y|^{-C_2'} + 1)(|x' - y'|^{-C_2'} + 1) \\ & \times (|x - x'|^{p\cdot \mathrm{e}^{-C_2 T}} + |y - y'|^{p\cdot \mathrm{e}^{-C_2 T}}),\end{aligned}$$

where C_2 is independent of T.

By Kolmogorov's criterion (cf. [3, Lemma 1.1]), we may further choose $T_4 \leqslant T_1$ small enough such that

$$\{(x, y) \in \mathbb{R}^{2d} : x \neq y\} \ni (x, y) \mapsto R_{-1}^{T_4}(x, y) = \sup_{t\in[0,T_4]} |Y_t(x) - Y_t(y)|^{-1}$$

admits a continuous version. In particular, this proves that for almost all $\omega \in \Omega$, $x \mapsto Y_t(x, \omega)$ is one-to-one for all $t \in [0, T_4]$.

As for the onto property, it follows along the same lines as in [5,9,14] by Lemma 3.4. We omit the details. Finally, we can use the shifting time method as used in Section 2 to prove the homeomorphism property for any large time T (see [14] for more details).

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Appendix

The following result is taken from [10, Theorem 6].

Theorem A.1. *Let M be a locally square integrable martingale such that $\Delta M > -1$ a.s. If*

$$\mathbb{E}\left[\exp\left\{\frac{1}{2} < M^c, M^c >_T + < M^d, M^d >_T\right\}\right] < \infty,$$

where M^c and M^d are continuous and purely discontinuous martingale parts of M, then $\mathcal{E}(M)$ is a martingale on $[0, T]$, where $\mathcal{E}(M)$ is the Doléans-Dade exponential given by

$$\mathcal{E}(M)_t := \exp\left\{M_t - \frac{1}{2}[M^c, M^c]_t\right\} \times \prod_{0<s\leqslant t}(1 + \Delta M_s)\mathrm{e}^{-\Delta M_s}.$$

The following lemma is a slight generalization of [12, Proposition 1.12, p. 476].

Lemma A.2. *Let $L(u)$ satisfy (2). Then for any $C, T > 0$*

$$\mathbb{E}\exp\left\{\sum_{0<s\leqslant T} CL(p_s)^2 \cdot 1_{\mathbb{U}_0}(p_s)\right\} = \exp\left\{T\int_{\mathbb{U}_0}(e^{CL(u)^2} - 1)\nu(\mathrm{d}u)\right\} < +\infty.$$

Proof. First of all, since $(\mathbb{U}, \mathcal{U}, \nu)$ is a σ-finite measure space, we may choose a sequence of sets $\{\mathbb{U}_n\}_{n\in\mathbb{N}}$ with finite measure $\nu(\mathbb{U}_n) < +\infty$ for every $n \in \mathbb{N}$, and such that $\mathbb{U}_n \uparrow \mathbb{U}_0$.

Define $L_n(u) := 1_{\mathbb{U}_n}(u) \cdot L(u)$ and

$$Z_t := \sum_{0<s\leqslant t} CL_n(p_s)^2.$$

Putting $\phi(t) := \mathbb{E}\exp\{Z_t\}$, we have by $L(u) \leqslant 1$

$$\phi(t) \leqslant \mathbb{E}\exp\left\{C\sum_{0<s\leqslant t} 1_{\mathbb{U}_n}(p_s)\right\} = \mathbb{E}\exp\{CN_p((0, t], \mathbb{U}_n)\} < +\infty.$$

Noticing that Z_t is a purely discontinuous process, we may write

$$\begin{aligned}\exp\{Z_t\} &= 1 + \sum_{0<s\leqslant t}(\exp\{Z_s\} - \exp\{Z_{s-}\}) \\ &= 1 + \sum_{0<s\leqslant t}\exp\{Z_{s-}\}(\exp\{CL_n(p_s)^2\} - 1).\end{aligned}$$

Taking expectations and by the master formula in [12, p. 475], we then have

$$\begin{aligned}\phi(t) &= 1 + \mathbb{E}\left[\int_0^t \exp\{Z_{s-}\}\int_{\mathbb{U}_0}(\exp\{CL_n(u)^2\} - 1)\nu(\mathrm{d}u)\mathrm{d}s\right] \\ &= 1 + \int_0^t \phi(s-)\left[\int_{\mathbb{U}_0}(\exp\{CL_n(u)^2\} - 1)\nu(\mathrm{d}u)\right]\mathrm{d}s.\end{aligned}$$

It follows that ϕ is continuous, so that

$$\phi(t) = \exp\left\{t\int_{\mathbb{U}_0}(\mathrm{e}^{CL_n(u)^2} - 1)\nu(\mathrm{d}u)\right\},$$

which then gives the formula by letting $n \to \infty$ and Fatou's lemma. $\square$

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